Fundamentals on Base Stations in Urban Cellular Networks: From the Perspective of Algebraic Topology

Ying Chen, Rongpeng Li, Zhifeng Zhao, and Honggang Zhang

***Abstract*—In recent decades, deployments of cellular networks have been going through an unprecedented expansion. In this regard, it is beneficial to acquire profound knowledge of cellular networks from the view of topology so that prominent network performances can be achieved by means of appropriate placements of base stations (BSs). In our researches, practical location data of BSs in eight representative cities are processed with classical algebraic geometric instruments, including α-shapes, Betti numbers, and Euler characteristics. At first, a fractal nature is validated in urban BS topology from both perspectives of Betti numbers and Hurst exponents. Furthermore, log-normal distribution is affirmed to provide the optimal fitness to the Euler characteristics of real urban BS deployments.**

***Index Terms*—Base stations, telecommunication network topology, algebra, data analysis.**

I. Introduction

A. Motivations

PROMPTED by the significance of base station (BS) deployment issues, substantial researchers have been working on the relevant subjects over decades [1]–[3]. However, the majority of the related documents studied BS deployments only by means of analyzing BS density distributions.

In our earlier works [4], several principal concepts in algebraic geometry field, i.e., α-Shapes, Betti numbers, and Euler characteristics [5], have been merged into the analyses of BS topology of 12 countries around the world, and meaningful topological features have been discovered, including the fractal property and log-normal distribution of the Euler characteristics [4]. As we know, the same rules hold for the whole system are not necessarily correct for a part of the system, and vice versa, such as the AdaBoost algorithm in deep learning [6] and massive MIMO (multiple-input multiple-output) in wireless systems [7]. On the other hand, from the perspectives of practical applications and engineering, the BS topological features in urban cellular networks may be more important than those in the scale of countries because a large number of users tend to live in capital cities. All of those concerns trigger our serious reflections on such a problem: will the topological features in national cellular networks still hold in the scale of dense cities?

Manuscript received November 11, 2018; accepted December 14, 2018. Date of publication December 21, 2018; date of current version April 9, 2019. This work was supported by National Natural Science Foundation of China under Grant 61701439 and Grant 61731002. The associate editor coordinating the review of this paper and approving it for publication was J. Coon. *(Corresponding author: Zhifeng Zhao.)*

The authors are with the College of Information Science and Electronic Engineering, Zhejiang University, Hangzhou 310058, China (e-mail: 21631088chen_ying@zju.edu.cn; lirongpeng@zju.edu.cn; zhaozf@zju.edu.cn; honggangzhang@zju.edu.cn).

Digital Object Identifier 10.1109/LWC.2018.2889041

B. Algebraic Geometric Tools

In the algebraic geometric field, firstly, α-Shapes [5] of a discrete point set are consistent with its intuitive shapes. In general, an α-Shape is a manifold which can be constructed based on the set of points and a given scale parameter α, thus a finite series of manifolds is obtained as α varies from 0 to $+\infty$. Secondly, Betti numbers [8] describe the topological information of a manifold, i.e., an α-Shape, by holes. Specifically, in the case of 3-dimensional space, a 0-dimensional hole (expressed by β_0) is an independent component in the α-Shape. A 1-dimensional hole (expressed by β_1) means the induced tunnel after plotting an edge between two directed or indirected connected points in the α-Shape. A 2-dimensional hole (expressed by β_2) is a cavity or void in the α-Shape enclosed by a 2-dimensional surface. Thirdly, Euler characteristics [5] capture the topological features of a manifold through global statistical properties. According to the Euler-Poincare Formula, Euler characteristic is equivalent to the alternating sum of Betti numbers. Due to the space limitation, interesting readers could refer to [4] to find the details of these three topological notions and their relationships.

Virtually, these seemingly abstract algebraic geometric tools are closely connected with various practical systems including wireless networking scenarios, and they have been applied in a vast number of networks. For instance, α-Shapes and Betti numbers have been incorporated in the cosmic Web [5]. Moreover, Betti numbers, or so-called persistent homology, have provided an alternative perspective for coverage problems in wireless sensor networks [9]. In addition, Euler characteristic has been utilized to study the impacts of dynamic topology of connections on the performance of recurrent artificial neural networks [10].

C. Contributions

To be specific, this letter searches for topological features in BS deployments, which transcend geographical, historic, or culture distinctions, and provide valuable guidance for designers of cellular networks. For this purpose, this letter offers two novel contributions as listed below:

- Firstly, a fractal nature is uncovered in urban BS configurations based on Betti numbers and Hurst exponents;
- Secondly, log-normal distribution is confirmed to match the Euler characteristics of real urban BS location data with the best fitting among all candidate distributions.

Actually, the results in this letter could be applied in a plenty of works for the study of complicated cellular networks. On one hand, fractal features could facilitate more sophisticated stochastic geometry models of cellular networks and enhance the performance evaluation of cellular networks [11], [12]. On the other hand, log-normal distribution of the Euler characteristics is a completely valuable discovery, and we believe it

TABLE I
BASIC INFORMATION OF 8 SELECTED CITIES

City		Area ($/\mathrm{km}^2$)	No. of BSs	Density of BSs ($1/\mathrm{km}^2$)	Hurst
Europe	Warsaw	512	7327	14.31	0.96912
	London	1577.3	21110	13.38	0.99591
	Munich	310.43	6837	22.02	0.93708
	Paris	12012	161753	13.47	0.93917
Asia	Seoul	605.77	9202	15.19	0.94064
	Tokyo	2188	46755	21.37	0.90492
	Beijing	16410	138620	8.45	0.98367
	Mumbai	603.4	13900	23.04	0.91981

will provide fruitful guidance on the analysis and design of BS deployments in the future.

The organization of this letter is arranged as follows. The actual BS location data are briefly introduced in Section II. The fractal phenomenon in BS deployments for either Asian or European cities is presented in Section III. Log-normal distribution is affirmed to offer the best fitting for the Euler characteristics of all the eight cities in Section IV. Lastly, conclusions are summarized in Section V.

II. DATASET DESCRIPTION AND PROCESSING PROCEDURES

Massive real data downloaded from *OpenCellID* community (https://community.opencellid.org/) [13], a platform to provide BS's location data all over the world, are analyzed to guarantee the validity of our results. Substantial BS location data of eight cities, including four representative cities in Asia (i.e., Seoul, Tokyo, Beijing, and Mumbai) and Europe (i.e., Warsaw, London, Munich, and Paris) respectively, have been collected from this platform. The basic information of these eight cities is provided in Table I.

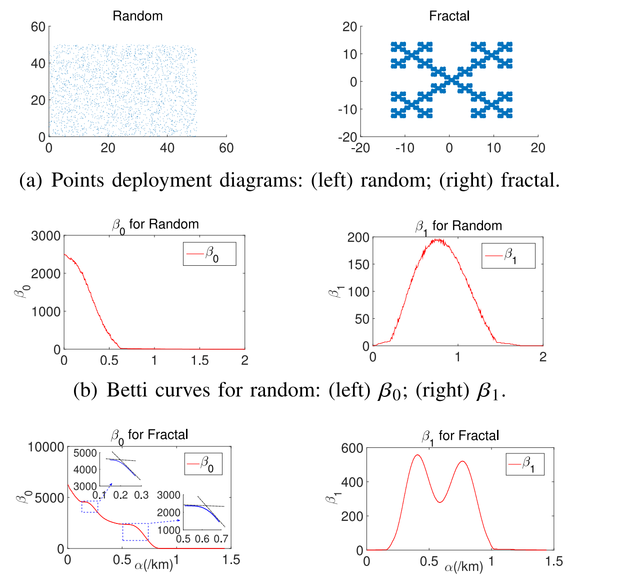
The data processing procedure from the original BS location data to the results in this letter is given as follows. Firstly, a collection of BS nodes is abstracted into a discrete point set in the 2-dimensional plane. Then an α-Shape [5] can be constructed given a scale parameter α. As α grows from 0 to $+\infty$, a finite series of α-Shapes is obtained. Secondly, each of the α-Shapes has the corresponding values of Betti numbers [8] and Euler characteristic [5]. Thus a series of Betti numbers and Euler characteristics can be achieved. Thirdly, taking α as the horizontal axis and Betti numbers as the vertical axes, Betti curves in this letter can be depicted. Lastly, PDF (probability density function) fitting curves can also be obtained according to the series of Euler characteristics.

III. FRACTAL NATURE IN CELLULAR NETWORKS TOPOLOGY

As a vital property of complex networks, the fractal nature has been revealed in a plenty of wireless networking scenarios [14]–[18]. In terms of Betti numbers and Hurst exponents, the fractal feature in BS topology is verified in this section.

A. Fractal Features Based on Betti Numbers

Random and fractal point distributions can be distinguished by their Betti curves as depicted in Fig. 1 (for details see [8]).



(a) Points deployment diagrams: (left) random; (right) fractal.

(b) Betti curves for random: (left) β_0; (right) β_1.

(c) Betti curves for fractal: (left) β_0; (right) β_1.

Fig. 1. Comparisons between the Betti curves of random and fractal point distributions.

Fig. 1(a) displays the practical point diagrams for both cases. For the left part, the horizontal and vertical coordinates of each point are randomly designated according to Poisson point distribution (PPP), whereas the fractal point deployment in the right part is realized by hierarchical partitions of the area, which brings the most principal aspect of fractal features, i.e., self-similarity [19], into the point distribution.

Fig. 1(b) illustrates the Betti curves for the random pattern, while Fig. 1(c) for the fractal pattern. Instead of the clearly monotonous decease in the β_0 curve and the single peak in the β_1 curve in the random situation, the fractal nature is manifested by the distinctive features of multiple ripples and peaks in the β_0 and β_1 curves, respectively [8], where a ripple is formed due to the distinct slope change as highlighted by the amplified blue subgraph in Fig. 1(c).

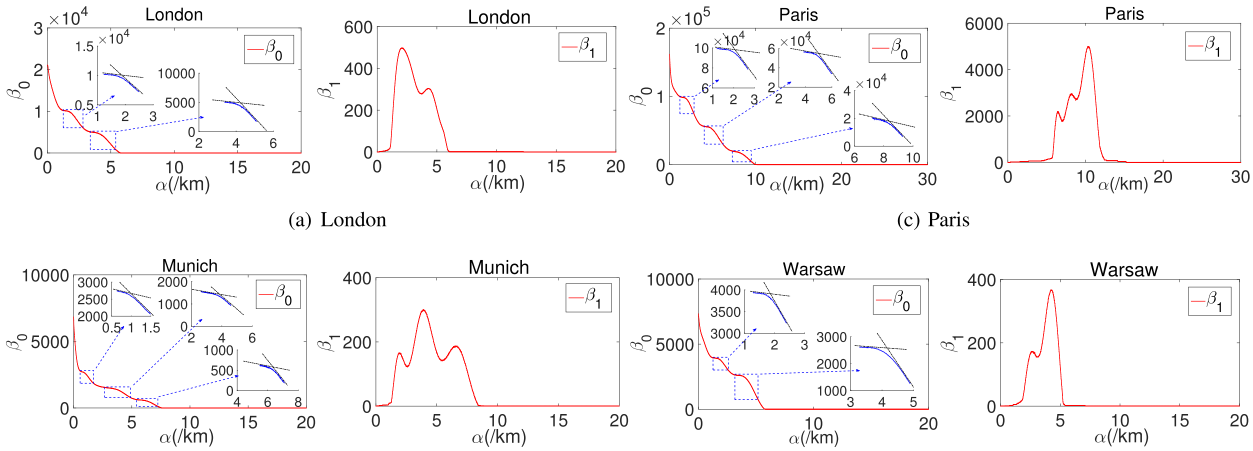
In summary, the fractal property can be characterized by the multiple ripples and peaks of the Betti curves [8]. In our works, the fractal property in BS topology is verified in Fig. 2 for European cities and Fig. 3 for Asian ones, respectively. Beyond geographical constraints, it is extremely astonishing to find out the consistent fractal nature for all the aforementioned eight cities.

Moreover, the entirely different positions of the multiple ripples and peaks can be evaluated on the basis of the crossover point of two straight lines with quite different gradients, as shown by the β_0 curves in Fig. 2 and Fig. 3. As listed in Table II, it is clear that the peaks always come after the corresponding ripples, which makes sense because of the larger size of loops than that of components.

B. Fractal Features Based on Hurst Exponents

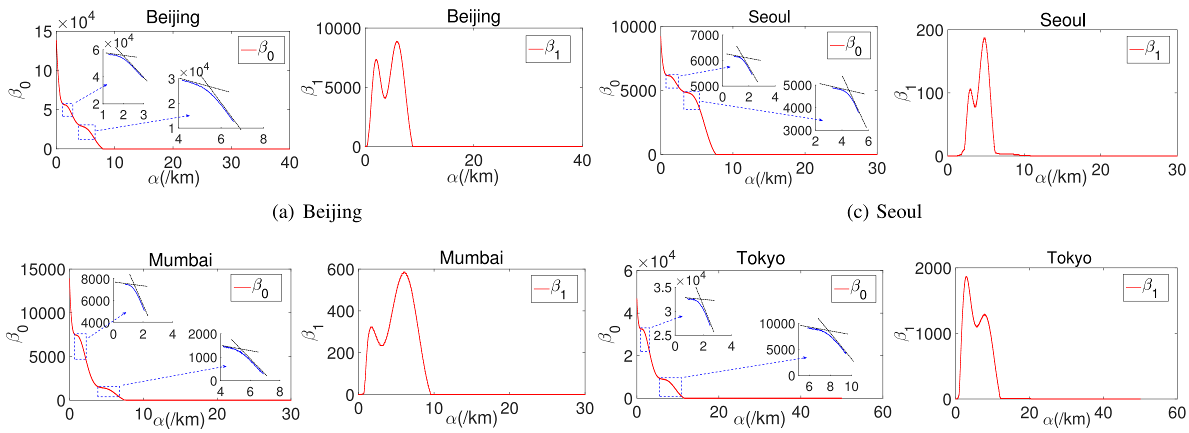
The Hurst exponent between [0, 1] is widely used as an evaluation index of the fractal property of data series, and the indication of fractality increases gradually as a Hurst exponent approaches to 1 [20].

For verifying the fractal nature in BS topology, the Hurst exponents of all the eight cities are listed in Table I.



(a) London (b) Munich (c) Paris (d) Warsaw

Fig. 2. Betti curves of the practical BS deployments in European cities.



(a) Beijing (b) Mumbai (c) Seoul (d) Tokyo

Fig. 3. Betti curves of the practical BS deployments in Asian cities.

TABLE II

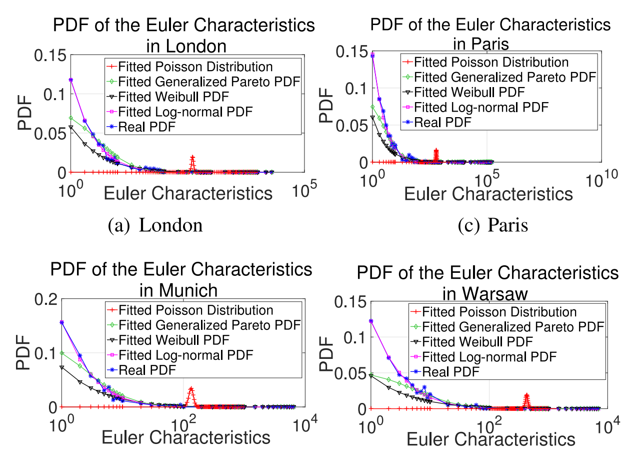
POSITIONS OF THE RIPPLES AND PEAKS IN THE BETTI CURVES

City	Index	β0	β1	City	Index	β0	β1
Warsaw	1st	1.880	2.620	Seoul	1st	1.530	4.515
	2nd	4.180	4.290		2nd	2.970	4.815
London	1st	1.840	2.090	Tokyo	1st	1.725	2.850
	2nd	4.240	4.360		2nd	7.725	7.800
Munich	1st	0.990	1.940	Beijing	1st	1.920	2.060
	2nd	3.820	3.920		2nd	5.360	5.800
	3rd	6.320	6.570				
Paris	1st	2.055	6.405	Mumbai	1st	1.350	1.725
	2nd	4.890	8.175		2nd	5.310	6.090
	3rd	8.340	10.330				

TABLE III

CANDIDATE DISTRIBUTIONS AND THEIR PDF EXPRESSIONS

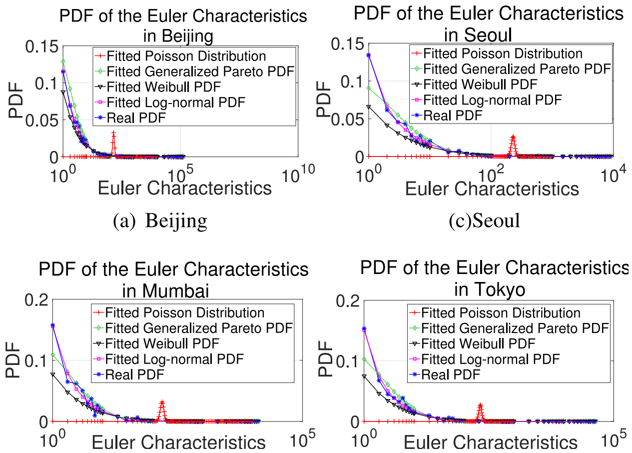
Distributions	Poisson	Weibull	Lognormal	Generalized Pareto(GP)
PDF	$\frac{\lambda^k}{k!}e^{-\lambda}$	$pqx^{q-1}e^{-px^q}$	$\frac{1}{\sqrt{2\pi}nx}e^{-\frac{(\ln x-m)^2}{2n^2}}$	$\frac{1}{b}(1+\frac{a}{b}x)^{-(1+\frac{1}{a})}$



(a) London (b) Munich (c) Paris (d) Warsaw

Fig. 4. Comparisons between the practical PDF and the fitted ones for the Euler characteristics of BS locations in European cities.

The definition of a data series and the computation for the Hurst exponents are illuminated as follows. Firstly, a center point is selected randomly among the BS location data of a city, and a circle is drawn given a random value of radius. Secondly, the distance between each of the points within the circle and the center point is computed, and these distance values form the data series for computing the Hurst exponent. Thirdly, a Hurst exponent can be calculated according to the R/S method [21]. Lastly, the above steps are operated a hundred times with different center points and radii, and the average value is the final Hurst exponent. The fractal nature



(a) Beijing (c)Seoul

(b) Mumbai (d) Tokyo

Fig. 5. Comparisons between the practical PDF and the fitted ones for the Euler characteristics of BS locations in Asian cities.

TABLE IV
RMSE Between Each Candidate Distribution and the Practical One

City \ Distributions		Possion ($\times 10^{-3}$)	Lognormal ($\times 10^{-3}$)	Generalized _Pareto ($\times 10^{-3}$)	Weibull ($\times 10^{-3}$)
Europe	Warsaw	2.433	0.355	1.039	1.189
	London	1.378	0.161	0.402	0.525
	Munich	3.150	0.189	0.839	1.265
	Paris	0.032	0.005	0.014	0.025
Asia	Seoul	2.302	0.098	0.582	0.841
	Tokyo	1.109	0.085	0.286	0.418
	Beijing	0.604	0.083	0.133	0.129
	Mumbai	2.136	0.138	0.546	0.817

is evidently affirmed because all the Hurst exponents turn out to be very close to 1.

IV. Log-Normal Distribution of the Euler Characteristics

Euler characteristics can be calculated from Betti numbers according to the Euler-Poincare Formula [5]. Since a clear heavy-tailed property is demonstrated in the PDFs of the Euler Characteristics, three classical heavy-tailed statistical distributions and widely-used Poisson distribution are selected as the candidates to match the PDFs. The candidates and their PDF formulas are given in Table III.

The real PDF and the fitted ones are presented in Fig. 4 for European cities and in Fig. 5 for Asian ones, respectively. Moreover, the root mean square error (RMSE) between the real PDF and every candidate is listed in Table IV. It is obvious that the RMSE between log-normal distribution and the real PDF is almost one-order of magnitude smaller than the others in the same row. As a result, an astounding but well-grounded conclusion can be drawn as follows: regardless of geographical boundaries, culture differences, and historical limitations, the Euler characteristics of both Asian and European cities entirely conform to log-normal distribution.

V. Conclusion and Future Works

In this letter, several algebraic geometric tools, namely, α-Shapes, Betti numbers, and Euler characteristics, have been utilized in the discovery of inherent topological essences in BS deployments for Asian and European cities. Firstly, a fractal nature has been revealed in BS topology according to both Betti numbers and Hurst exponents. Moreover, it has been proved that log-normal distribution provides the best match for the PDFs of the Euler characteristics of the real BS location data in cellular networks among the classical candidate distributions.

However, regardless of the topological discoveries above, some thought-provoking problems still need to be solved. For example, what are the influential factors for the positions of the ripples and peaks in the Betti curves? What is the intrinsic meaning of the number of ripples or peaks? All of these issues will be investigated in our future works.

References

[1] J. Kibiłda, B. Galkin, and L. A. DaSilva, “Modelling multi-operator base station deployment patterns in cellular networks,” *IEEE Trans. Mobile Comput.*, vol. 15, no. 12, pp. 3087–3099, Dec. 2016.

[2] Y. Zhou *et al.*, “Large-scale spatial distribution identification of base stations in cellular networks,” *IEEE Access*, vol. 3, pp. 2987–2999, 2016.

[3] Z. Zhao, M. Li, R. Li, and Y. Zhou, “Temporal-spatial distribution nature of traffic and base stations in cellular networks,” *IET Commun.*, vol. 11, no. 16, pp. 2410–2416, 2017.

[4] Y. Chen, R. Li, Z. Zhao, and H. Zhang, “Study on base station topology in cellular networks: Take advantage of alpha shapes, Betti numbers, and Euler characteristics,” *arXiv:1808.07356v1*, Aug. 2018.

[5] R. V. D. Weygaert *et al.*, *Alpha, Betti and the Megaparsec Universe: On the Topology of the Cosmic Web*. Heidelberg, Germany: Springer, 2011.

[6] J. Zhu, S. Rosset, H. Zou, and T. Hastie, “ Multi-class AdaBoost,” *Stat. Interface*, vol. 2, no. 3, pp. 349–360, 2006.

[7] E. G. Larsson, O. Edfors, F. Tufvesson, and T. L. Marzetta, “Massive MIMO for next generation wireless systems,” *IEEE Commun. Mag.*, vol. 52, no. 2, pp. 186–195, Feb. 2014.

[8] P. Pranav *et al.*, “The topology of the cosmic Web in terms of persistent Betti numbers,” *Monthly Notices Roy. Astronomical Soc.*, vol. 4, no. 465, pp. 4281–4310, Mar. 2017.

[9] V. D. Silva and R. Ghrist, “Coverage in sensor networks via persistent homology,” *Algebraic Geometric Topol.*, vol. 7, no. 1, pp. 339–358, Apr. 2007.

[10] P. Masulli and A. E. P. Villa, “The topology of the directed clique complex as a network invariant,” *Springerplus*, vol. 5, no. 388, pp. 1–12, 2016.

[11] R. Li, Z. Zhao, Y. Zhong, C. Qi, and H. Zhang, “The stochastic geometry analyses of cellular networks with α-stable self-similarity,” *IEEE Trans. Commun.*, to be published, doi: 10.1109/TCOMM.2018.2883099.

[12] X. Ge, X. Tian, Y. Qiu, G. Mao, and T. Han, “Small-cell networks with fractal coverage characteristics,” *IEEE Trans. Commun.*, vol. 66, no. 11, pp. 5457–5469, Nov. 2018.

[13] M. Ulm, P. Widhalm, and N. Brändle, “Characterization of mobile phone localization errors with OpenCellID data,” in *Proc. Int. Conf. Adv. Logistics Transp.*, Valenciennes, France, May 2015, pp. 100–104.

[14] C. Yuan, Z. Zhao, R. Li, M. Li, and H. Zhang, “The emergence of scaling law, fractal patterns and small-world in wireless networks,” *IEEE Access*, vol. 5, pp. 3121–3130, 2017.

[15] X. Ge *et al.*, “Wireless fractal cellular networks,” *IEEE Wireless Commun.*, vol. 23, no. 5, pp. 110–119, Oct. 2016.

[16] Y. Hao *et al.*, “Wireless fractal ultra-dense cellular networks,” *Sensors*, vol. 17, no. 4, p. E841, 2017.

[17] S. H. Strogatz, “Complex systems: Romanesque networks,” *Nature*, vol. 433, no. 7024, pp. 365–366, 2005.

[18] C. Song, S. Havlin, and H. A. Makse, “Self-similarity of complex networks,” *Nature*, vol. 433, no. 7024, pp. 392–395, 2005.

[19] D. Benhaiem, M. Joyce, and B. Marcos, “Self-similarity and stable clustering in a family of scale-free cosmologies,” *Monthly Notices Roy. Astronomical Soc.*, vol. 443, no. 3, pp. 2126–2153, 2014.

[20] T. Gneiting and M. Schlather, “Stochastic models which separate fractal dimension and Hurst effect,” *Soc. Ind. Appl. Math. Rev.*, vol. 46, no. 2, pp. 269–282, 2001.

[21] M. Fernández-Martínez, M. A. Sánchez-Granero, J. E. T. Segovia, and I. M. Román-Sánchez, “An accurate algorithm to calculate the Hurst exponent of self-similar processes,” *Phys. Lett. A*, vol. 378, nos. 32–33, pp. 2355–2362, 2014.